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AN APPLICATION OF THE MULTIVARIATE EXTENDED POISSON
DISTRIBUTION IN 2 X 2 CONTINGENCY TABLES


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| 16. ABSTRACT <p>In a series of 2x2 contingency tables, suppose that $n(>1)$, the total number of individuals, varies according to a Poisson distribution. In these situations, to test the independence of the two characteristics involved, the likelihood ratio test is developed which also makes use of the additional information that the distribution of n is known to be Poisson. The unconditional distribution of the cell frequencies is obtained, and the first approximates for the iteration procedure to obtain the maximum likelihood estimates of the parameters of the distribution under the hypothesis of independence of the two characteristics are given.</p> | | | |
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AN APPLICATION OF THE MULTIVARIATE
EXTENDED POISSON DISTRIBUTION IN 2×2
CONTINGENCY TABLES

by

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1. Introduction

The various methods for making a combined test of independence of two characteristics in the data consisting of a series of 2×2 tables obtained under different situations have been described by Cochran [1]. Let n , the total number of individuals in a 2×2 table, be divided as follows:

| | | |
|----------|----------|----------|
| x_{00} | x_{01} | $x_{0.}$ |
| x_{10} | x_{11} | $x_{1.}$ |
| $x_{.0}$ | $x_{.1}$ | n |

If π_{00} , π_{01} , π_{10} and π_{11} (such that $\pi_{00} + \pi_{01} + \pi_{10} + \pi_{11} = 1$)

are the corresponding probabilities, the probability distribution of (x_{00}, x_{01}, x_{10}) will be multinomial with parameters n , π_{00} , π_{01} and π_{10} .

Now let us consider a situation where n (>1) varies according to a Poisson distribution. For example, the number of accidents observed per week on a certain high-way may be divided into a 2×2 contingency table according to major and minor accidents and day-time and night-time accidents.

Applications of perhaps greater interest to a space scientists might involve thunderstorm accompanied by rain versus thunderstorm without rain; thunderstorm within, say, a ten mile radius of the observer versus thunderstorm at a specific point, and other similar atmospheric phenomena. Here we may assume the Poisson distribution for the number of occurrences. In such situations, we can make use of the additional information that the

distribution of n is known. To test the independence in these situations, in this paper, the likelihood ratio test is developed.

2. Unconditional Distribution of the Cell Frequencies (X_{00}, X_{01}, X_{10}) .

For fixed n , the probability distribution of (X_{00}, X_{01}, X_{10}) will be given by a multinomial distribution with parameters n , π_{00} , π_{01} and π_{10} . The probability generating function of (X_{00}, X_{01}, X_{10}) in terms of Z_1, Z_2 and Z_3 will be given by

$$P(Z_1, Z_2, Z_3 | n) = (1 - \pi_{00} - \pi_{01} + \pi_{10} + \pi_{00}Z_1 + \pi_{01}Z_2 + \pi_{10}Z_3)^n$$

If we assume that n is a random variable with truncated Poisson distribution

$$P_n = \frac{e^{-\lambda} \cdot \lambda^n}{n! (1 - e^{-\lambda})}, \quad n = 1, 2, \dots; \lambda > 0$$

then the unconditional probability generating function of (X_{00}, X_{01}, X_{10}) will be given by (Khatri [3])

$$P(Z_1, Z_2, Z_3) = \sum_{n=1}^{\infty} P(Z_1, Z_2, Z_3 | n) \cdot \frac{e^{-\lambda} \cdot \lambda^n}{n! (1 - e^{-\lambda})}.$$

Hence the unconditional distribution of (X_{00}, X_{01}, X_{10}) will become (Cohen [2]) $f(X_{00}, X_{01}, X_{10}; m_1, m_2, m_3, \theta)$

$$= \begin{cases} 1 - \theta & \text{for } X_{00} = X_{01} = X_{10} = 0 \\ \theta \cdot \frac{e^{-(m_1 + m_2 + m_3)} m_1^{X_{00}} m_2^{X_{01}} m_3^{X_{10}}}{X_{00}! X_{01}! X_{10}! [1 - e^{-(m_1 + m_2 + m_3)}]}, & \text{otherwise} \end{cases} \dots \quad (2.1)$$

Where $m_1 = \lambda \pi_{00}$

$$m_2 = \lambda \pi_{01}$$

$$m_3 = \lambda \pi_{10}$$

and
$$\theta = \frac{1 - \exp[-m_1 - m_2 - m_3]}{1 - \exp(-\lambda)}$$

This may be considered as an extended multivariate Poisson distribution.

3. Formulation of Hypotheses:

Taking $\pi_{.0}$ and $\pi_{0.}$ as the marginal probabilities of given two characteristics, under the assumption of independence, $\pi_{00} = \pi_{.0} \times \pi_{0.}$, $\pi_{01} = \pi_{.1} \times \pi_{0.}$ and $\pi_{10} = \pi_{.0} \times \pi_{1.}$

| | | |
|------------|------------|------------|
| π_{00} | π_{01} | $\pi_{0.}$ |
| π_{10} | π_{11} | $\pi_{1.}$ |
| $\pi_{.0}$ | $\pi_{.1}$ | 1.0 |

Thus we have,

H_0 : Cell-probabilities can be written as the product of the marginal probabilities as shown above

and H_1 : Cell probabilities cannot be written as the product of the marginal probabilities

Under H_0 , the probability distribution (2.1) becomes (after having the relation $\lambda = (m_3 + m_1)(m_2 + m_1) / m_1$)

$$\frac{1-\exp(-\alpha)}{\exp(\beta)-\exp(-\alpha)} \quad , \quad \text{for } X_{00} = X_{01} = X_{10} = 0$$

$$f(X_{00}, X_{01}, X_{10}; m_1, m_2, m_3) =$$

$$\frac{m_1^{X_{00}} m_2^{X_{01}} m_3^{X_{10}}}{X_{00}! X_{01}! X_{10}! [\exp(\beta) - \exp(-\alpha)]} \quad , \quad \text{otherwise}$$

.... (3.1)

Where $\alpha = m_2 m_3 / m_1$ and $\beta = m_1 + m_2 + m_3$. Here α and β are introduced only for convenience; our interest is still lying only in m_1, m_2 and m_3 .

4. Maximum Likelihood Estimates of the Parameter of distribution (2.1).

We have a series of groups of observations and each group is divided into a 2 x 2 contingency table as described before. Let f_{ijk} = frequency for $X_{00} = i$, $X_{01} = j$ and $X_{10} = k$. Then the likelihood function based on (2.1) will be given as

$$L = (1-\theta)^{f_{000}} \prod_{\substack{i \ j \ k \\ i+j+k>0}} \left(\frac{\theta \exp(-\beta) m_1^i m_2^j m_3^k}{i! j! k! [1-\exp(-\beta)]} \right)^{f_{ijk}}$$

Writing $C = \log i! + \log j! + \log k!$, we get $\log L = f_{000} \log(1-\theta) + \sum_{\substack{i \ j \ k \\ i+j+k>0}} f_{ijk}$

$$\{ \log \theta - \beta + i \log m_1 + j \log m_2 + k \log m_3 - C - \log [1 - \exp(-\beta)] \}$$

Differentiating $\log L$ partially with respect to θ , m_1 , m_2 and m_3 and writing $A = 1 - \exp(-\beta)$, we obtain

$$\frac{\partial \log L}{\partial \theta} = -\frac{f_{000}}{1-\theta} + \frac{\sum_{i,j,k} f_{ijk}}{\theta}$$

$$\frac{\partial \log L}{\partial m_1} = \sum_{i,j,k} f_{ijk} \left[\frac{i}{m_1} - \frac{1}{A} \right]$$

(4.1)

$$\frac{\partial \log L}{\partial m_2} = \sum_{i,j,k} f_{ijk} \left[\frac{j}{m_2} - \frac{1}{A} \right]$$

$$\frac{\partial \log L}{\partial m_3} = \sum_{i,j,k} f_{ijk} \left[\frac{k}{m_3} - \frac{1}{A} \right]$$

Equating the system (4.1) to zero, we get

$$\hat{\theta} = \frac{N-f_{000}}{N}$$

$$\hat{m}_1 = N \bar{X}_{00} \hat{A} / (N-f_{000})$$

(4.2)

$$\hat{m}_2 = N \bar{X}_{01} \hat{A} / (N-f_{000})$$

$$\hat{m}_3 = N \bar{X}_{10} \hat{A} / (N-f_{000})$$

Where $N = \sum_{i,j,k} f_{ijk}$, the number of 2 x 2 tables, $\hat{A} = 1 - \exp(-\hat{\beta})$ and $\hat{\beta}$

$= \hat{m}_1 + \hat{m}_2 + \hat{m}_3$. From last three equations of (4.2), we get

$$\hat{m}_1 = \hat{m}_2 \bar{X}_{00} / \bar{X}_{01}$$

(4.3)

$$\hat{m}_3 = \hat{m}_2 \bar{X}_{10} / \bar{X}_{01}$$

Substituting (4.3) in the last equation of (4.2) we get

$$1 - \exp\{-\hat{m}_2(\bar{X}_{00} + \bar{X}_{01} + \bar{X}_{10}) / \bar{X}_{01}\} = (N-f_{000}) \cdot \hat{m}_2 / N \cdot \bar{X}_{01} \quad \dots \quad (4.4)$$

Expanding the exponential term, step by step, first upto linear term in \hat{m}_2 , then upto quadratic term in \hat{m}_2 and so on, we shall get successive improved estimates of m_2 . This method is to be repeated until two successive values of \hat{m}_2 become nearly equal. Finally \hat{m}_1 and \hat{m}_3 will be obtained.

5. Maximum Likelihood Estimates of the parameters of distribution (2.1) under the hypothesis of independence.

Under H_0 , the log likelihood will be given by

$$\begin{aligned} \log L = f_{000} \log[1 - \exp(-\alpha)] - \{ \log[\exp(\beta) - \exp(-\alpha)] \} \sum_i \sum_j \sum_k f_{ijk} \\ + \sum_i \sum_j \sum_k f_{ijk} [i \log m_1 + j \log m_2 + k \log m_3 - c], \\ i+j+k > 0 \end{aligned}$$

where $c = \log i! + \log j! + \log k!$ (from (3.1))

Differentiating partially with respect to m_1 , m_2 and m_3 we get

$$\begin{aligned} \frac{\partial \log L}{\partial m_1} &= \frac{f_{000}}{1 - \exp(-\alpha)} \{-\exp(-\alpha) m_2 m_3 / m_1^2\} - \left\{ \frac{\exp(\beta) - \exp(-\alpha) m_2 m_3 / m_1^2}{\exp(\beta) - \exp(-\alpha)} \right\} N + N \bar{X}_{00} / m_1 \\ \frac{\partial \log L}{\partial m_2} &= \frac{f_{000}}{1 - \exp(-\alpha)} \{\exp(-\alpha) m_3 / m_1\} - \left\{ \frac{\exp(\beta) + \exp(-\alpha) m_3 / m_1}{\exp(\beta) - \exp(-\alpha)} \right\} N + N \bar{X}_{01} / m_2 \\ \frac{\partial \log L}{\partial m_3} &= \frac{f_{000}}{1 - \exp(-\alpha)} \{\exp(-\alpha) m_2 / m_1\} - \frac{\exp(\beta) + \exp(-\alpha) m_2 / m_1}{\exp(\beta) - \exp(-\alpha)} N \\ &\quad + N \bar{X}_{10} / m_3 \dots \dots \dots (5.1) \end{aligned}$$

Let us define $\frac{\partial \log L}{\partial m_i} = g_i(m_1, m_2, m_3)$. If \hat{m}_1^0 , \hat{m}_2^0 and \hat{m}_3^0 are the first approximate values of the estimates of m_1 , m_2 and m_3 respectively, then according to the Taylor Series expansion, we get the following equations. (Expansion is considered only upto the first power of $(\hat{m}_i - \hat{m}_i^0)$).

$$\left. \begin{aligned} \sum_{i=1}^3 (\hat{m}_i - \hat{m}_i^0) \frac{\partial g_1(m_1, m_2, m_3)}{\partial m_i} \bigg|_{\hat{m}_1^0, \hat{m}_2^0, \hat{m}_3^0} &= -g_1(m_1^0, m_2^0, m_3^0) \\ \sum_{i=1}^3 (\hat{m}_i - \hat{m}_i^0) \frac{\partial g_2(m_1, m_2, m_3)}{\partial m_i} \bigg|_{\hat{m}_1^0, \hat{m}_2^0, \hat{m}_3^0} &= -g_2(m_1^0, m_2^0, m_3^0) \\ \sum_{i=1}^3 (\hat{m}_i - \hat{m}_i^0) \frac{\partial g_3(m_1, m_2, m_3)}{\partial m_i} \bigg|_{\hat{m}_1^0, \hat{m}_2^0, \hat{m}_3^0} &= -g_3(m_1^0, m_2^0, m_3^0) \end{aligned} \right\} \dots (5.2)$$

By iteration, we can solve these equations for \hat{m}_1, \hat{m}_2 , and \hat{m}_3 .

6. First Approximates under H_0 , of m_1, m_2 and m_3 by using moments of first order and zero cell frequency.

Under H_0 , the probability distribution of (X_{00}, X_{01}, X_{10}) is given by

$$f(X_{00}, X_{01}, X_{10}; m_1, m_2, m_3) = \begin{cases} [1 - \exp(-\alpha)] / [\exp(\beta) - \exp(-\alpha)] & \text{for } X_{00} = X_{01} = X_{10} = 0 \\ \frac{m_1^{X_{00}} m_2^{X_{01}} m_3^{X_{10}}}{X_{00}! X_{01}! X_{10}! [\exp(\beta) - \exp(-\alpha)]} & \text{otherwise} \end{cases}$$

Where $\alpha = m_2 m_3 / m_1$ and $\beta = m_1 + m_2 + m_3$

It can be shown that

$$\left. \begin{aligned} E(X_{00}) &= m_1 \exp(\beta) / [\exp(\beta) - \exp(-\alpha)] \\ E(X_{01}) &= m_2 \exp(\beta) / [\exp(\beta) - \exp(-\alpha)] \\ \text{and } E(X_{10}) &= m_3 \exp(\beta) / [\exp(\beta) - \exp(-\alpha)] \end{aligned} \right\} \dots \dots \dots (6.1)$$

Now equating the probability of $X_{00} = X_{01} = X_{10} = 0$ to the ratio of zero frequency to the total frequency in the sample, we get

$$f_{000} [\exp(\hat{\beta}) - \exp(-\hat{\alpha})] = N[1 - \exp(-\hat{\alpha})] \dots \dots \dots (6.2)$$

Equating (6.1) to the corresponding sample means, we get

$$\left. \begin{aligned} \bar{X}_{00} &= \hat{m}_1 \exp(\hat{\beta}) / [\exp(\hat{\beta}) - \exp(-\hat{\alpha})] \\ \bar{X}_{01} &= \hat{m}_2 \exp(\hat{\beta}) / [\exp(\hat{\beta}) - \exp(-\hat{\alpha})] \\ \bar{X}_{10} &= \hat{m}_3 \exp(\hat{\beta}) / [\exp(\hat{\beta}) - \exp(-\hat{\alpha})] \end{aligned} \right\} \dots\dots (6.3)$$

$$\left. \begin{aligned} \text{Hence, } \hat{m}_2 &= \bar{X}_{01} \hat{m}_1 / \bar{X}_{00} \\ \text{and } \hat{m}_3 &= \bar{X}_{10} \hat{m}_1 / \bar{X}_{00} \end{aligned} \right\} \dots (6.4)$$

Substituting $\exp(\hat{\beta}) - \exp(-\hat{\alpha}) = N[\exp(\hat{\beta}) - 1] / (N - f_{000})$ (from (6.2)) in the first equation of (6.3), we obtain

$$\hat{m}_1 + \hat{m}_2 + \hat{m}_3 = \log \bar{X}_{00} - \log \left[\bar{X}_{00} - \hat{m}_1 + \frac{f_{000}}{N} \hat{m}_1 \right] \dots (6.5)$$

Substituting (6.4) in (6.5), we get

$$\hat{m}_1 + \frac{\bar{X}_{00}}{\bar{X}_{00} + \bar{X}_{01} + \bar{X}_{10}} \cdot \log \left[1 - \frac{N - f_{000}}{N} \cdot \frac{\hat{m}_1}{\bar{X}_{00}} \right] = 0 \dots (6.6)$$

It can be seen that equations (6.4) and (6.6) are equivalent to the equations (4.3) and (4.4). Thus we can use the maximum likelihood estimates obtained in the general case as the first approximates for the iteration procedure for obtaining the maximum likelihood estimates under H_0 .

7. Likelihood Ratio Test.

Let $(\underline{X}_1, \underline{X}_2, \dots, \underline{X}_p)$ be a sample from the probability density function (2.1), where \underline{X}_i is 3-dimensional vector having the components $(X_{00i}, X_{01i}, X_{10i})$. Under H_0 , the p.d.f. (2.1) reduces to p.d.f. (3.1) containing only 3 parameters. Let Ω_4 be the 4-dimensional parameter space for which $m_1 > 0, m_2 > 0, m_3 > 0, \theta > 0$. Let ω_3 be the subset of Ω_4 for which H_0 is true. (H_0 : the hypothesis of independence of two given characteristics). Thus for this test the likelihood

ratio λ is given by

$$\lambda = \frac{\sup_{\omega_3} \text{likelihood function}}{\sup_{\Omega_4} \text{likelihood function}}$$

Where the estimates of the parameters given in (5) are used to obtain the numerator and those given in (4) are used to obtain the denominator.

For large number of groups (number of 2x2 tables), the distribution of $-2\log\lambda$ is approximately χ^2 with 1 d.f. if H_0 is true [Wilks (4)].

8. A Numerical Illustration:-

A random sample of size 31 was drawn from a Poisson population with mean $\lambda = 3.3333$. The following are the observations.

2,5,8,5,4,3,8,2,6,5,6,1, 2,5,3,6,4,3,3,2,5,2,4,6,2,3,1,2,4,4,4.

Each of the above observations was considered as the total size of a 2x2 contingency table. Thus, 31 tables were constructed. The marginal probabilities $\pi_{0\cdot}$ and $\pi_{\cdot 0}$ were selected to be 1/3 and 2/5 respectively.

This gave the following table under the hypothesis of independence.

| | | |
|-----------------------|-----------------------|----------------------|
| $\pi_{00} = 2/15$ | $\pi_{01} = 3/15$ | $\pi_{0\cdot} = 1/3$ |
| $\pi_{10} = 4/15$ | $\pi_{11} = 6/15$ | $\pi_{1\cdot} = 2/3$ |
| $\pi_{\cdot 0} = 2/5$ | $\pi_{\cdot 1} = 3/5$ | 1.0 |

The total size of a table was divided randomly into 4 cells according to the multinomial distribution with the parameters given by the above table. The following 2x2 tables were observed:

| Table number | x_{00} | x_{01} | x_{10} | x_{11} | Total size |
|--------------|----------|----------|----------|----------|------------|
| 1 | 1 | 0 | 1 | 0 | 2 |
| 2 | 0 | 0 | 1 | 4 | 5 |
| 3 | 1 | 3 | 3 | 1 | 8 |
| 4 | 1 | 1 | 2 | 1 | 5 |
| 5 | 1 | 1 | 0 | 2 | 4 |
| 6 | 0 | 2 | 0 | 1 | 3 |
| 7 | 2 | 2 | 1 | 3 | 8 |
| 8 | 0 | 1 | 1 | 0 | 2 |
| 9 | 1 | 1 | 3 | 1 | 6 |
| 10 | 0 | 1 | 1 | 3 | 5 |
| 11 | 0 | 2 | 1 | 3 | 6 |
| 12 | 0 | 0 | 0 | 1 | 1 |
| 13 | 0 | 0 | 1 | 1 | 2 |
| 14 | 0 | 2 | 1 | 2 | 5 |
| 15 | 1 | 0 | 0 | 2 | 3 |
| 16 | 1 | 1 | 1 | 3 | 6 |
| 17 | 0 | 1 | 2 | 1 | 4 |
| 18 | 0 | 1 | 1 | 1 | 3 |
| 19 | 2 | 0 | 0 | 1 | 3 |
| 20 | 0 | 0 | 1 | 1 | 2 |
| 21 | 1 | 0 | 3 | 1 | 5 |
| 22 | 0 | 1 | 0 | 1 | 2 |
| 23 | 1 | 0 | 1 | 2 | 4 |
| 24 | 1 | 2 | 2 | 1 | 6 |
| 25 | 1 | 0 | 0 | 1 | 2 |
| 26 | 0 | 1 | 1 | 1 | 3 |
| 27 | 0 | 0 | 0 | 1 | 1 |
| 28 | 0 | 1 | 1 | 0 | 2 |
| 29 | 0 | 1 | 2 | 1 | 4 |
| 30 | 0 | 1 | 2 | 1 | 4 |
| 31 | 1 | 1 | 1 | 1 | 4 |

In the space Ω , the maximum likelihood estimates of m_1 , m_2 , m_3 and θ were obtained using (4.2), (4.3) and (4.4). Using these as initial estimates of m_1 , m_2 and m_3 , the maximum likelihood estimates of the parameters under the

null hypothesis were obtained by carrying out the iteration procedure (5.2) on IBM 7090/7094 using Fortran IV. The solution converged to five places of decimals in 4 cycles.

In the space Ω , we obtained

$$\hat{m}_1 = 0.53610, \hat{m}_2 = 0.85146, \hat{m}_3 = 1.07221 \text{ and } \hat{\theta} = 0.93550$$

In the space ω , we obtained

$$\hat{m}_1 = 0.54197, \hat{m}_2 = 0.85561, \hat{m}_3 = 1.07796$$

These results gave

$$-2\log_e \lambda = 0.02422$$

Considering $-2\log \lambda$ as approximately χ^2 with 1 d.f., the value of $-2\log \lambda$ is not at all significant resulting in the acceptance of H_0 i.e. the hypothesis of independence.

Combining all the tables into one, χ^2 (usual test) was obtained as 0.4243. This value, too, is not significant. It should, however, be noted that the value of χ^2 obtained according our test is much lower as compared to the one obtained by combining all the tables. This suggests that the test developed here may be more efficient. However, further studies into the efficiency aspect are necessary before reaching a final conclusion.

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